BALANCED METRICS ON HOMOGENEOUS VECTOR BUNDLES

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Abstract. Let $E \to M$ be a holomorphic vector bundle over a compact Kähler manifold (M, ω) and let $E = E_1 \oplus \cdots \oplus E_m \to M$ be its decomposition into irreducible factors. Suppose that each E_j admits a ω-balanced metric in Donaldson-Wang terminology. In this paper we prove that E admits a unique ω-balanced metric if and only if $\frac{r_j}{N_j} = \frac{r_k}{N_k}$ for all $j, k = 1, \ldots, m$, where r_j denotes the rank of E_j and $N_j = \dim H^0(M, E_j)$. We apply our result to the case of homogeneous vector bundles over a rational homogeneous variety (M, ω) and we show the existence and rigidity of balanced Kähler embedding from (M, ω) into Grassmannians.

1. Introduction

Let $E \to M$ be a very ample holomorphic vector bundle of rank r with $\dim(H^0(M, E)) = N$. Choose a basis $\underline{s} = (s_1, \ldots, s_N)$ of $H^0(M, E)$, the space of global holomorphic sections of E, we denotes with $i_{\underline{s}} : M \to G(r, N)$ the Kodaira map associated to the basis $\underline{s}$ (see, e.g. [11]).

Consider the flat metric h_0 on the tautological bundle $\mathcal{T} \to G(r, N)$, i.e. $h_0(v, w) = w^* v$, and the dual metric $h_{Gr} = h_0^*$ on the quotient bundle $\mathcal{Q} = \mathcal{T}^*$. Let $\omega_{Gr} = P^* \omega_{FS}$ be the Kähler form on $G(r, N)$ pull-back of the Fubini–Study form $\omega_{FS} = \frac{i}{2} \partial \bar{\partial} \log(|z_0|^2 + \cdots + |z_{\widetilde{N}-1}|^2)$ via the Plücker embedding $P : G(r, N) \to \mathbb{C}P^{\widetilde{N}-1}$, with $\widetilde{N} = \binom{N}{r}$.

We can endow $E = i_{\underline{s}}^* \mathcal{Q}$ with the hermitian metric

$$h_{\underline{s}} = i_{\underline{s}}^* h_{Gr} \tag{1}$$

and define a L^2-product on $H^0(M, E)$ by the formula:

$$\langle \cdot, \cdot \rangle_{h_{\underline{s}}} = \frac{1}{V(M)} \int_M h_{\underline{s}}(\cdot, \cdot) \frac{\omega^n}{n!}, \tag{2}$$

where $\omega^n = \omega \wedge \cdots \wedge \omega$ and $V(M) = \int_M \frac{\omega^n}{n!}$.

Date: January 15, 2011.

2000 *Mathematics Subject Classification.* 53D05; 53C55; 58F06.

Key words and phrases. Kähler metrics; balanced metric; balanced basis; holomorphic maps into grassmannians; hermitian symmetric space.

Research partially supported by GNSAGA (INdAM) and MIUR of Italy.

An hermitian metric h on E is called *ω-balanced* if there exists a basis $\underline{s}$ of $H^0(M,E)$ such that $h = h_{\underline{s}} = i_{\underline{s}}^* h_{Gr}$ and

$$\langle s_j, s_k \rangle_{h_{\underline{s}}} = \frac{r}{N}\delta_{jk}, \quad j,k = 1,\ldots,N = \dim H^0(M,E). \tag{3}$$

The concept of balanced metrics on complex vector bundles was introduced by X. Wang [17] (see also [18]) following S. Donaldsons ideas introduced in [5] in order to characterize which manifolds admit a constant scalar curvature Kähler metric. While existence of ω-balanced metrics is still a very difficult and obscure problem, uniqueness had been proved by Loi–Mossa (cfr. Theorem 2). Study of balanced metrics is a very fruitful area of research both from mathematical and physical point of view (see, e.g., [2], [3], [4], [6], [8], [9] and [10]).

Regarding the existence and uniqueness of ω-balanced basis we have the following fundamental results (see next section for the definition of the Gieseker point T_E of E).

Theorem 1. *(X. Wang,* [18]*) The Gieseker point T_E is stable (in the GIT terminology) if and only if E admits a ω-balanced metric.*

Theorem 2. *(A. Loi–R. Mossa,* [11]*) Let E be a holomorphic vector bundle over a compact Kähler manifold (M,ω). If E admits a ω-balanced metric then the metric is unique.*

The following theorem is the main result of this paper (the "only if" part is already contained in Theorem 2 of [11] and we include it here for completeness).

Theorem 3. *Let (M,ω) a Kähler manifold and let $E_j \to M$ be a very ample vector bundles over M with* $\operatorname{rank} E_j = r_j$ *and* $\dim H^0(M,E_j) = N_j > 0$*, for $j = 1,\ldots,m$. Suppose that each E_j admits a ω-balanced metric. Then the vector bundle $E = \oplus_{j=1}^m E_j \to M$ admits a unique ω-balanced metric if and only if $\frac{r_j}{N_j} = \frac{r_k}{N_k}$ for all $j,k = 1,\ldots,m$.*

When (M,ω) is a rational homogeneous variety and $E \to M$ is an irreducible homogeneous vector bundle over M, L. Biliotti and A. Ghigi [1] proved that E admits a unique ω-balanced metric. Hence we immediately get the following result.

Corollary 4. *Let (M,ω) be a rational homogeneous variety and let $E_j \to M$ be irreducible homogeneous vector bundles over M with* $\operatorname{rank} E_j = r_j$ *and* $\dim H^0(M,E_j) = N_j > 0$*, for $j = 1,\ldots,m$. Then the homogeneous vector bundle $E = \oplus_{j=1}^m E_j \to M$ admits a unique homogeneous ω-balanced metric if and only if $\frac{r_j}{N_j} = \frac{r_k}{N_k}$ for all $j,k = 1,\ldots,m$.*

The paper contained other two sections. In the next one we prove Theorem 3. In the last section we apply our result to prove the existence and

rigidity of balanced maps of rational homogeneous varieties into grassmannians.

Acknowledgements: I wish to thank Prof. Alessandro Ghigi and Prof. Andrea Loi for various interesting and stimulating discussions.

2. Proof of Theorem 3

The Gieseker point T_E of a vector bundle E of rank r is the map

$$T_E : \bigwedge^r H^0(M,E) \to H^0(M, \det E)$$

which sends $s_1 \wedge \cdots \wedge s_r \in \bigwedge^r H^0(M,E)$ to the holomorphic section of $\det E$ defined by

$$T_E(s_1 \wedge \cdots \wedge s_r) : x \mapsto s_1(x) \wedge \cdots \wedge s_r(x).$$

The group $\mathrm{GL}(H^0(M,E))$ acts on $H^0(M,E)$, therefore we get an action also on $\bigwedge^r H^0(M,E)$ and on $\mathrm{Hom}(\bigwedge^r H^0(M,E), H^0(M,\det E))$. The actions are given by

$$V \cdot (s_1 \wedge \cdots \wedge s_r) = V s_1 \wedge \cdots \wedge V s_r$$

and

$$(V \cdot T)(s_1 \wedge \cdots \wedge s_r) = T(V \cdot (s_1 \wedge \cdots \wedge s_r)).$$

where $T \in \mathrm{Hom}\left(\bigwedge^r H^0(M,E), H^0(M,\det E)\right)$ and $V \in \mathrm{GL}\left(H^0(M,E)\right)$.

Recall also that if G is a reductive group acting linearly on a vector space V then an element v of V is called *stable* if Gv is closed in V and the stabilizer of v inside G is finite.

Proof of Theorem 3 Without loss of generality we can assume $m = 2$. Hence assume that E is a direct sum of two holomorphic vector bundles $E_1, E_2 \to M$ with $\operatorname{rank} E_j = r_j$ and $\dim H^0(M,E_j) = N_j > 0$, $j = 1,2$.

Suppose first that $\frac{N_1}{r_1} = \frac{N_2}{r_2}$. Let $\underline{s}^j = \{s^j_1, \ldots, s^j_{N_j}\}$ be the basis of $H^0(M,E_j)$ for $j = 1,2$. Then, the assumption $\frac{r_1}{N_1} = \frac{r_1}{N_2}$, readily implies that the basis

$$\underline{s} = ((s^1_1, 0), \ldots, (s^1_{N_1}, 0), (0, s^2_1), \ldots, (0, s^2_{N_2})) \tag{4}$$

is a homogeneous ω-balanced basis for $E_1 \oplus E_2$. Therefore $h_{\underline{s}} = i^*_{\underline{s}} h_{Gr}$ is the desired homogeneous balanced metric on $E_1 \oplus E_2$ which is unique by Theorem 2.

Viceversa if $\frac{N_1}{r_1} \neq \frac{N_2}{r_2}$ we claim that the Gieseker point of E is not stable and by Wang's Theorem 1 the bundle E can not admit a ω-balanced metric. In order to prove this consider the basis $\underline{s} = \{s_1, \ldots, s_{N_1+N_2}\}$ of $H^0(M,E)$ such that $\{s_1, \ldots, s_{N_1}\}$ is a basis of $H^0(M, E_1 \oplus \{0\})$ and

$\{s_{N_1+1}, \dots, s_{N_1+N_2}\}$ is a basis of $H^0(M, \{0\} \oplus E_2)$. Suppose that $\frac{N_1}{r_1} > \frac{N_2}{r_2}$. Consider the 1-parameter subgroup of $SL(N_1 + N_2)$

$$t \mapsto g(t) = \mathrm{diag}(\underbrace{t^{-N_2}, \dots, t^{-N_2}}_{N_1 \text{ times}}, t^{N_1}, \dots, t^{N_1}),$$

where the action on the elements of the basis $\underline{s}$ is

$$g(t)s_j = \begin{cases} t^{-N_2} s_j & \text{if } j \leq N_1 \\ t^{N_1} s_j & \text{otherwise.} \end{cases}$$

Observe that the section $x \mapsto s_{j_1}(x) \wedge \cdots \wedge s_{j_r}(x)$ (with $j_1 < j_2 < \cdots < j_r$), were $r = r_1 + r_2$, is different from zero only when $j_{r_1} \leq N_1 < j_{r_1+1}$. So the action of $g(t)$ on the Gieseker point is given by:

$$g(t)T_E(s_{j_1} \wedge \cdots \wedge s_{j_r}) =$$
$$= \begin{cases} t^{r_2 N_1 - r_1 N_2} T_E(s_{j_1} \wedge \cdots \wedge s_{j_r}) & \text{if } j_{r_1} \leq N_1 < j_{r_1+1} \\ 0 & \text{otherwise} \end{cases}$$

Since $\frac{N_1}{r_1} > \frac{N_2}{r_2}$ this yields

$$\lim_{t \to 0} g(t) T_E \equiv 0.$$

and hence T_E is not stable with respect to the reductive group $\mathrm{GL}(H^0(M, E) = \mathrm{GL}(N_1 + N_2)$. □

3. Balanced maps of rational homogeneous varieties into grassmannians

Given a Kähler manifold (M, ω) we call $f : M \to G(r, N)$ a balanced Kähler embedding if it is a Kähler embedding with respect to the Kähler form ω_{Gr} on $G(r, N)$ and it is of the form $f = i_{\underline{s}}$, where $\underline{s}$ is a ω-balanced basis for $H^0(M, E)$ (i.e. the condition (3) is satisfied), with $E = f^* \mathcal{Q}$.

Theorem 5. *Let (M, ω) be a rational homogeneous variety and let $E \to M$ be an homogeneous vector bundle over M with $\omega \in c_1(E)$. Suppose that E is direct sum of irreducible homogeneous vector bundles $E_1, \dots, E_m \to M$ with* $\mathrm{rank}\, E_j = r_j$ *and* $\dim H^0(M, E_j) = N_j > 0$, $j = 1, \dots, m$. *Then there exists a unique (up to a unitary transformations of $G(r, N)$) balanced Kähler embedding $f : M \to G(r, N)$ such that $f^* Q = E$ if and only if $\frac{r_j}{N_j} = \frac{r_k}{N_k}$ for all $j, k = 1, \dots, m$.*

Proof. Theorem 4 tell us that there exists a balanced embedding if and only if is satisfied the condition $\frac{r_1}{N_1} = \cdots = \frac{r_m}{N_m}$. Suppose that $f : M \to G(r, N)$ is the balanced Kähler embedding. Let g an element of G the group of isometries of M, then the map $f \circ g : M \to G(r, N)$ is again a balanced Kähler embedding, so by Theorem 2 we get $(f \circ g)^* h_{Gr} = f^* h_{Gr}$. Therefore the Kähler form $i^* \omega_{Gr}$ is homogeneous, indeed

$$g^*(f^* \omega_{Gr}) = (f \circ g)^* \omega_{Gr} = \mathrm{Ric}((f \circ g)^* h_{Gr}) = \mathrm{Ric}(f^* h_{Gr}) = f^* \omega_{Gr}.$$

Since over a rational homogeneous variety every cohomology class contains exactly one invariant form, then $i^*\omega_{Gr} = \omega$. $\square$

As a corollary of the previous result we get:

Corollary 6. *Let (M, ω) be an hermitian symmetric space of compact type (HSSCT) of dimension r and with $\dim H^0(M, TM) = N$. Assume that (M, g) is the product of irreducible hermitian symmetric spaces of compact type $(M_1, \omega_1), \ldots, (M_m, \omega_m)$. Then (M, ω) admit a balanced Kähler embedding in $G(r, N)$ if and only if $\frac{r_j}{N_j} = \frac{r_k}{N_k}$ for all $j, k = 1, \ldots, m$, where $\dim(M_j) = r_j$ and $\dim H^0(M, TM_j) = N_j > 0$.*

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